

# ON THE MAXIMAL DEGREE OF THE KHOVANOV HOMOLOGY

KEIJI TAGAMI

**ABSTRACT.** It is known that the maximal homological degree of the Khovanov homology of a knot gives a lower bound of the minimal positive crossing number of the knot. In this paper, we show that the maximal homological degree of the Khovanov homology of a cabling of a knot gives a lower bound of the minimal positive crossing number of the knot.

## 1. INTRODUCTION

In [4], for each oriented link  $L$ , Khovanov defined a graded chain complex whose graded Euler characteristic is equal to the Jones polynomial of  $L$ . Its homology is a link invariant and called the Khovanov homology. Throughout this paper, we denote the homological degree  $i$  term of the rational Khovanov homology of a link  $L$  by  $KH^i(L)$ .

Let  $L$  be an oriented link. By  $i_{\max}(L)$ , we denote the maximal homological degree of the Khovanov homology of  $L$ , and by  $c_+(L)$ , we denote the minimal number of the positive crossings of diagrams of  $L$ . Note that  $i_{\max}(L)$  is not negative. In fact, any link has nonzero Khovanov homology in degree zero because the Lee homology [6] is not trivial in homological degree zero. Then, it is known that  $i_{\max}(L)$  gives a lower bound of  $c_+(L)$  (Proposition 2.1). From this fact, it seems that the Khovanov homology estimates the positivity of links.

Stošić [9] showed that  $i_{\max}(T_{2k,2kt})$  is  $2k^2t$ , where  $T_{p,q}$  is the positive  $(p,q)$ -torus link. By using the same method as Stošić, the author [10] proved that  $i_{\max}(T_{2k+1,(2k+1)t})$  is  $2k(k+1)t$ . The author also computed the maximal degree for a cabling of any knot.

In this paper, we give some properties of the maximal degree of the Khovanov homology. In particular, we show that the maximal homological degree of the Khovanov homology of a cabling of a knot gives a lower bound of the minimal positive crossing number of the knot as follows:

**Theorem 1.1.** *Let  $K$  be an oriented knot. Denote the  $(p,q)$ -cabling of  $K$  by  $K(p,q)$ . For any positive integers  $p$  and  $t$ , we assume that each component of  $K(p,pt)$  has the orientation induced by  $K$ , that is, each component of  $K(p,pt)$  is homologous to  $K$  in the tubular neighborhood of  $K$ . Then, if  $t \leq 2c_+(K)$ , we have the following inequality:*

$$\frac{i_{\max}(K(p,pt))}{p^2} \leq c_+(K).$$

In particular, we obtain

$$\frac{i_{\max}(K(p, 0))}{p^2} \leq c_+(K).$$

In many cases,  $p^2 c_+(K)$  is not greater than the minimal positive crossing number of  $(p, 0)$ -cabling of  $K$ . Hence the inequality of Theorem 1.1 is possibly stronger than that of Proposition 2.1.

Note that there are some works on the crossing numbers of cable links (for example, see [3], [5, Problem 1.68] and [8]).

This paper is organized as follows: In Section 2, we give the proof of Theorem 1.1. In Appendix, we introduce some properties of the maximal degree of the Khovanov homology.

We refer some informations (knot names, values of knot invariants and so on) in [7]. Throughout this paper, we use the same definition and notation of the Khovanov homology as in [10, p.2848-p.2850]. In particular, for a link diagram  $D$  of a link  $L$ , we denote the unnormalized Khovanov homology by  $H^i(D)$ , that is,

$$KH^i(L) = H^{i+c_-(D)}(D),$$

where  $c_-(D)$  is the number of the negative crossings of  $D$ .

## 2. THE POSITIVITY OF KNOTS AND THE MAXIMAL DEGREE OF THE KHOVANOV HOMOLOGY

In this section, we give some estimates on the minimal positive crossing numbers of knots. In particular, we prove Theorem 1.1.

For any oriented link  $L$ , define

$$\begin{aligned} i_{\max}(L) &:= \max\{i \in \mathbf{Z} \mid KH^i(L) \neq 0\}, \\ i_{\min}(L) &:= \min\{i \in \mathbf{Z} \mid KH^i(L) \neq 0\}, \\ c_{\pm}(L) &:= \min\{c_{\pm}(D) \mid D \text{ is a diagram of } L\}, \end{aligned}$$

where  $c_+(D)$  ( $c_-(D)$ ) is the number of the positive (negative) crossings of  $D$ . A link  $L$  is *positive* (*negative*) if  $c_-(L) = 0$  ( $c_+(L) = 0$ ). The following is an immediate consequence of the definition of the Khovanov homology.

**Proposition 2.1** (cf. [12, Proposition 2.2]). *For any oriented link  $L$ , we have  $i_{\max}(L) \leq c_+(L)$  and  $-i_{\min}(L) \leq c_-(L)$ .*

Unfortunately, Proposition 2.1 is not sufficient to determine whether a given link is negative (or positive) (for example, see Example 2.3). In [10], the author computed  $i_{\max}$  for a cable link. By the computation, we obtain new estimates of the minimal positive crossing numbers of knots (Theorem 1.1).

*Proof of Theorem 1.1.* Let  $D$  be a diagram of  $K$  with  $c_+(K)$  positive crossings. Let  $c_-(D)$  be the number of the negative crossings of  $D$ .

Suppose that  $p = 2k$ . By [10, Lemma 4.3 (1)] (with  $j = 2k$  and  $m = 0$ ), we have

$$H^i(D^0(2k, 2k(n + c_+(K) - c_-(D)) + 2k)) = 0$$

for  $i > (2k)^2(c_+(K) + c_-(D))$  and  $0 \leq n < c_+(K) + c_+(D)$ .<sup>1</sup> Here, the diagram  $D^0(2k, 2k(n + c_+(K) - c_-(D)) + 2k)$  is introduced in [10, Definitions 3.1, 3.9 and

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<sup>1</sup>In [10, Lemma 4.3], we put  $l := c_+(D) + c_+(D)$  and  $f := c_+(D) - c_-(D)$ . In our setting,  $c_+(D) = c_+(K)$ .

4.1, and Figures 4 and 8]. Moreover  $D^0(2k, 2k(n + c_+(K) - c_-(D)) + 2k)$  is equal to  $D(2k, 2k(n + c_+(K) - c_-(D) + 1))$ , where  $D(p, q)$  is a diagram of  $K(p, q)$  introduced in [10, Definition 4.1 and Figures 7 and 9]. Since the diagram  $D(2k, 2k(n + c_+(K) - c_-(D) + 1))$  has  $(2k)^2 c_-(D)$  negative crossings, by putting  $t = n + c_+(K) - c_-(D) + 1$ , we obtain

$$\mathrm{KH}^i(K(2k, 2kt)) = 0$$

for  $i > (2k)^2 c_+(K)$  and  $c_+(K) - c_-(D) + 1 \leq t \leq 2c_+(K)$ . In particular, we have  $i_{\max}(K(2k, 2kt)) \leq (2k)^2 c_+(K)$ . By using the negative first Reidemeister move repeatedly, we can take as large a  $c_-(D)$  as we want. Hence, for  $-\infty < t \leq 2c_+(K)$ , we have

$$i_{\max}(K(2k, 2kt)) \leq (2k)^2 c_+(K).$$

Suppose that  $p = 2k + 1$ . By [10, Lemma 4.3 (2)] and the same discussion as above, we obtain

$$i_{\max}(K(2k + 1, (2k + 1)t)) \leq (2k + 1)^2 c_+(K)$$

for  $-\infty < t \leq 2c_+(K)$ .  $\square$

**Corollary 2.2.** *If  $K$  is a negative knot, then  $i_{\max}(K(p, 0))$  is zero for any positive integer  $p$ .*

**Example 2.3.** We see that  $i_{\max}(8_{21}) = i_{\max}(\overline{9_{45}}) = i_{\max}(\overline{9_{46}}) = 0$  (see [7]). Hence, the inequality in Proposition 2.1 cannot determine whether these knots are negative. However, by “The Mathematica Package KnotTheory” [2], we have

$$i_{\max}(8_{21}(2, 0)) = i_{\max}(\overline{9_{45}}(2, 0)) = 2, \quad i_{\max}(\overline{9_{46}}(2, 0)) = 4.$$

By Theorem 1.1, these knots are not negative.

**Remark 2.4.** Let  $K$  be an oriented knot. Then,  $\mathrm{KH}^i(K(p, pt)) = 0$  if  $i$ ,  $n$  and  $p$  satisfy one of the following conditions (see also Figure 1):

- (1)  $i > p^2 c_+(K)$  and  $t \leq 2c_+(K)$ ,
- (2)  $p = 2k$  for some  $k > 0$ ,  $i > 2k^2(t - 2c_+(K)) + p^2 c_+(K)$  and  $t > 2c_+(K)$ ,
- (3)  $p = 2k + 1$  for some  $k > 0$ ,  $i > 2k(k + 1)(t - 2c_+(K)) + p^2 c_+(K)$  and  $t > 2c_+(K)$ .

Condition (1) follows from Theorem 1.1, and (2) and (3) follow from [10, Lemma 4.3].

**Question 2.5.** *For any non-negative knot  $K$ , are there some  $p > 0$  and  $t \leq 2c_+(K)$  such that  $i_{\max}(K(p, pt)) > 0$ ?*

**Question 2.6.** *For any knot  $K$ , does the following hold?*

$$i_{\max}(K) \leq \frac{i_{\max}(K(p, 2pi_{\max}(K)))}{p^2} \leq c_+(K).$$

Note that the last inequality holds by Theorem 1.1.

## APPENDIX A. APPENDIX: OTHER PROPERTIES OF THE KHOVANOV HOMOLOGY

In this section, we introduce other properties of the maximal degree of the Khovanov homology.

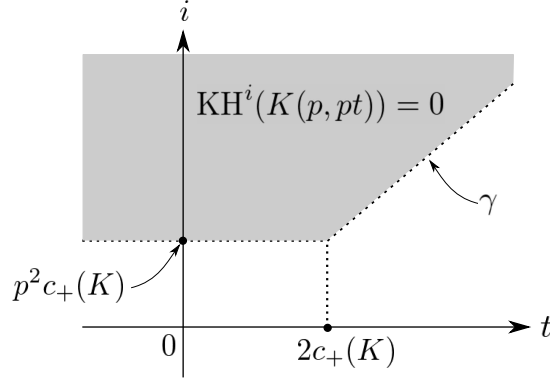


FIGURE 1.  $\text{KH}^i(K(p, pt)) = 0$  in the gray area. The slope of the line  $\gamma$  is  $2k^2$  if  $p = 2k$  and  $2k(k+1)$  if  $p = 2k+1$ .

**A.1.  $i_{\max}$  versus  $c_+$ .** By the following result, the accuracy of the inequality in Proposition 2.1 does not depend on the choice of orientations of links.

**Proposition A.1.1.** *For any oriented link  $L$ ,  $c_+(L) - i_{\max}(L)$  does not depend on the orientation of  $L$ .*

*Proof.* Let  $K$  be a component of  $L$ . Let  $L'$  be the link obtained from  $L$  by reversing the orientation of  $K$ . Khovanov [4, Proposition 28] proved that

$$\text{KH}^i(L') = \text{KH}^{i+2lk(L \setminus K, K)}(L).$$

Hence we have  $i_{\max}(L) = i_{\max}(L') + 2lk(L \setminus K, K)$ . On the other hand, we have  $c_+(L) = c_+(L') + 2lk(L \setminus K, K)$ . These imply this proposition.  $\square$

**Example A.1.2.** Let  $T_{p,q}$  be the positive  $(p, q)$ -torus link. For positive integers  $k$  and  $t$ , we have  $c_+(T_{2k, 2kt}) = 2kt(2k-1)$ . Indeed, the standard diagram of  $T_{2k, 2kt}$  has  $2kt(2k-1)$  positive crossings. Moreover, each 2-component sublink of  $T_{2k, 2kt}$  is  $T_{2, 2t}$  and has at least  $2t$  positive crossings. Since the link  $T_{2k, 2kt}$  has  $k(2k-1)$  2-component sublinks, we obtain  $c_+(T_{2k, 2kt}) \geq 2kt(2k-1)$ . On the other hand, Stojić [9, Theorem 3] showed that  $i_{\max}(T_{2k, 2kt}) = 2k^2t$ . Hence, we have  $c_+(T_{2k, 2kt}) - i_{\max}(T_{2k, 2kt}) = 2kt(k-1)$ .

Let  $T'_{2k, 2kt}$  be the link obtained from  $T_{2k, 2kt}$  by reversing the orientations of exactly  $k$  components. Similarly, we have  $c_+(T'_{2k, 2kt}) = 2kt(k-1)$  and  $i_{\max}(T'_{2k, 2kt}) = 0$ . Hence we have  $c_+(T'_{2k, 2kt}) - i_{\max}(T'_{2k, 2kt}) = 2kt(k-1)$ .

**Corollary A.1.3.** *For any positive integer  $N$ , there exists some oriented link  $L$  such that  $c_+(L) - i_{\max}(L) > N$ .*

*Proof.* By Example A.1.2, we have  $c_+(T_{4, 4N}) - i_{\max}(T_{4, 4N}) > N$ .  $\square$

**A.2. Additivity of  $i_{\max}$ .** We prove the additivity of the maximal degree of the Khovanov homology.

**Proposition A.2.1.** *For any oriented knots  $K_1$  and  $K_2$ , we have*

$$i_{\max}(K_1 \# K_2) = i_{\max}(K_1 \sqcup K_2) = i_{\max}(K_1) + i_{\max}(K_2),$$

where  $K_1 \sqcup K_2$  is the disjoint union of the knots and  $K_1 \# K_2$  is the connected sum of the knots.

*Proof.* By [4, Proposition 33], we obtain

$$\mathrm{KH}^i(K_1 \sqcup K_2) \cong \bigoplus_{p+q=i} \mathrm{KH}^p(K_1) \otimes \mathrm{KH}^q(K_2).$$

Note that our coefficient ring is the rational number field  $\mathbf{Q}$ . Hence we have  $i_{\max}(K_1 \sqcup K_2) = i_{\max}(K_1) + i_{\max}(K_2)$ .

In [4, Proposition 34], the following exact sequence was introduced:

$$\begin{aligned} \cdots \rightarrow \mathrm{KH}^i(K_1 \# K_2) &\rightarrow \mathrm{KH}^i(K_1 \sqcup K_2) \rightarrow \mathrm{KH}^i(K_1 \# K_2) \\ &\rightarrow \mathrm{KH}^{i+1}(K_1 \# K_2) \rightarrow \mathrm{KH}^{i+1}(K_1 \sqcup K_2) \rightarrow \mathrm{KH}^{i+1}(K_1 \# K_2) \\ &\rightarrow \mathrm{KH}^{i+2}(K_1 \# K_2) \rightarrow \mathrm{KH}^{i+2}(K_1 \sqcup K_2) \rightarrow \cdots \end{aligned}$$

Here, we forget the quantum grading of the Khovanov homology. By the first line, we have  $\mathrm{KH}^i(K_1 \# K_2) \neq 0$  if  $\mathrm{KH}^i(K_1 \sqcup K_2) \neq 0$ . Hence we obtain  $i_{\max}(K_1 \# K_2) \geq i_{\max}(K_1 \sqcup K_2)$ .

Put  $i_0 = i_{\max}(K_1 \sqcup K_2)$ . Then  $\mathrm{KH}^{i_0+1}(K_1 \sqcup K_2) = \mathrm{KH}^{i_0+2}(K_1 \sqcup K_2) = 0$ . By the second and third lines, we have

$$\mathrm{KH}^{i_0+1}(K_1 \# K_2) = \mathrm{KH}^{i_0+2}(K_1 \# K_2).$$

By repeating this process, for any  $l \in \mathbf{Z}_{>0}$ , we obtain

$$\mathrm{KH}^{i_0+1}(K_1 \# K_2) = \mathrm{KH}^{i_0+l}(K_1 \# K_2).$$

Since  $\mathrm{KH}^{i_0+l}(K_1 \# K_2) = 0$  for sufficiently large  $l$ , we have

$$\mathrm{KH}^{i_0+l}(K_1 \# K_2) = 0,$$

for any  $l \in \mathbf{Z}_{>0}$ . This implies that  $i_0 \geq i_{\max}(K_1 \# K_2)$ . Hence we obtain  $i_{\max}(K_1 \# K_2) = i_{\max}(K_1 \sqcup K_2) = i_{\max}(K_1) + i_{\max}(K_2)$ .  $\square$

**A.3. Almost positive knots.** A link diagram is *almost positive (negative)* if it has exactly one negative (positive) crossing. A link is *almost positive (negative)* if it is not a positive (negative) knot and has an almost positive (negative) diagram. Then we have the following.

**Corollary A.3.1.** *For any almost negative link  $L$ , we have  $i_{\max}(L) = 0$ .*

*Proof.* By Proposition 2.1, we have  $i_{\max}(L) = 0$  or  $1$ . For contradiction, assume that  $i_{\max}(L) = 1$ . Then, any almost negative diagram  $D$  of  $L$  satisfies  $i_{\max}(L) = c_+(D) = 1$ . In [4, Proposition 36], Khovanov proved that  $i_{\max}(L) = c_+(D)$  if and only if the diagram  $D$  is +adequate (for the definition of +adequate diagrams, for example, see [4, Definition 3]). However, any reduced almost negative link diagram is not +adequate. Hence,  $i_{\max}(L) = 0$ .  $\square$

**Remark A.3.2.** *The Khovanov homology and the Rasmussen invariant for an almost positive link are studied in [1] and [11].*

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY, TOKYO UNIVERSITY  
OF SCIENCE, NODA, CHIBA, 278-8510, JAPAN

*E-mail address:* tagami\_keiji@ma.noda.tus.ac.jp